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ON THE NORMALIZER OF A SUBGROUP OF A FINITE GROUP AND THE CAYLEY EMBEDDING

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1. Introduction

In this paper we study some properties of the normalizer of a subgroup of a finite group, and in particular, obtain some conditions under which the normalizer of a nilpotent subgroup contains the subgroup properly. If G is a nilpotent group and H is a proper subgroup, then it is an elementary fact that H is properly contained in its normalizer in G. We are considering here the dual situation. Recall that a subgroup H of a group G is called a *T.I. set* (trivial intersection set) if for $g \in G$, either $H^g = H$, or $H^g \cap H = (e)$. We prove

Proposition 1. *Let G be a finite group and H be a proper, nilpotent subgroup of G. Assume that H is a T.I. set. Then we have that* $H < N_G(H)$.

Proof. Suppose if possible that $H = N_G(H)$. Then since H is a T.I. set, it follows that G is a *Frobenius group* with complement H (see for example, [7, Proposition 8.2, p. 59]). Now H is nilpotent and therefore solvable. Using the results of H. Zassenhaus on the classification of solvable Frobenius complements (see, for example, [7, Proposition 18.2, p. 196]), it follows that all the Sylow subgroups of H are cyclic, and that H has a subgroup K such that H/K is isomorphic to Sym(4). Since by hypothesis, H is nilpotent, it follows that H/K is nilpotent. Thus it follows that Sym(4) is nilpotent, a contradiction. Hence, we have that $H = N_G(H)$. □

Example. Let $G := \mathrm{Sym}(3) \times \mathbb{Z}_3$. Take H to be a Sylow 2-subgroup of G. Then H

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is a T.I. set in G. Further H is contained in a cyclic subgroup of G of order 6. So, clearly $H<N_G(H)$.

If G is a finite group of composite order, then it is easy to see that there is at least one proper subgroup H of G such that $H<N_G(H)$. On the other hand, if G is a finite, solvable group then it is well known that G has always a nilpotent self-normalizing subgroup called a *Carter subgroup* after R.W. Carter who first obtained this result [3]; further any two Carter subgroups of a solvable group are conjugate to each other. For a non-solvable group, the existence of even at least one nilpotent, self-normalizing subgroup cannot be always be guaranteed. For example, in the non-solvable group A_5 every nilpotent subgroup is properly contained in its normalizer. However, if we consider a maximal, nilpotent subgroup H of a non-solvable group G, then it is easy to show that $N_G(H)=H$ and further it follows readily using a deep result of Thompson [10], that any two such maximal, nilpotent subgroups of G are conjugate. Non-solvable groups with nilpotent, maximal subgroups have been classified completely by Bauman [1], and Rose [8].

Next, we consider a (nilpotent) group G of order n and embed it in the symmetric group Sym(n) by the Cayley mapping (the right regular representation). We prove in Section 3 some properties of the normalizer of G (in its embedding) in Sym(n). These results which we obtain, are consequences of the following theorem which is of independent interest. First, we recall some standard definitions. Let G be a finite group and Π a finite set of primes. A subgroup H of G is called a *Π-group* if the order $|H|$ of H is divisible only the primes in Π; H is called a Π'-group if $|H|$ is not divisible by any prime in Π. A *normal Π-complement* of a Π-subgroup H of a finite group G is a Π'-group K such that $G=HK$, $K \lhd G$ and $H\cap K=(e)$.

Theorem 2. *Let G be a Π-group of order n where Π is a finite set of primes. Embed G into S_n by the Cayley mapping (right regular representation): $g\mapsto \binom{x}{xg}$. In this embedding, let X be a subgroup of S_n such that:*

$$G<X\leq S_n \tag{*}$$

Then G cannot have a normal Π-complement in X.

We use standard group theoretic terminology and notation as in Huppert [5]. We emphasise that failure to keep in mind the following standard notation may lead to a confusion: $H\leq G$ denotes that H is a subgroup of G, and $H<G$ indicates that H is a subgroup of G which is not equal to G. All groups considered here are finite.

2. Proof of Theorem 2

Suppose if possible that the group G has a normal Π-complement in X. Then we have $X=GH$ where H is a Π'-group. Now the group X operates on the set G by right multiplication and this action is necessarily transitive. Let o be some fixed 'point' of the set G. Let X_o be the isotropy group of the point o, that is $X_o := \{x \in X : o^x = o\}$ where o^x denotes the image of o under the action of the element x. Since the action of X on the set G is transitive, it follows that $|G| = [X : X_o]$. So we have, $|X_o| = |X|/|G| = |H|$ since H is a normal Π-complement of G in X. Now we claim:

$$\bigcap_{x \in X} x^{-1} X_o x = \text{(identity)}. \tag{2.1}$$

We prove (2.1) as follows. let u belong to $\bigcap x^{-1} X_o x$ where x ranges over the group X. Then we have that $u = x^{-1} \upsilon x$ for all $x \in X$ and some $\upsilon \in X_o$ depending on x. Now this implies that $xu = \upsilon x$ and so $o^{xu} = o^{\upsilon x} = o^x$ since $\upsilon \in X_0$. Since the action of the group X on the set G is transitive, it follows that every element of the set G can be expressed in the form o^x for some $x \in X$ (recall that o is some chosen element of the set G). Now $o^{xu} = o^x$ implies that u 'fixes' every point of the set G. However, the permutation action of X on the set G is a faithful action. Hence it follows that u is the identity, proving (2.1).

Now consider the composite of the following homomorphisms:

$$X_o \to X \to X/H \simeq G$$

Here, the first homomorphism is a natural embedding of a subgroup into a group containing it. Since $|X_o| = |H|$ and H is a Π-complement of G, we have that $|X_o|$ and $|G|$ are co-prime. Therefore, X_o must be equal to H. This however, contradicts (2.1) since for any $x \in X$, we have that $x^{-1} X_o x = x^{-1} H x = H$, as H is a normal subgroup of G. Hence, it follows that G cannot have a normal Π-complement in X. □

3. Normalizer of a group in the Cayley embedding

We now describe some results which are consequences of Theorem 2. The following theorem was proved by Bhattacharya [2].

Theorem 3. *Let G be an abelian group of prime power order satisfying the hypothesis of Theorem* 2. *Then we have that $G < N_X(G)$.*

Proof. Clearly, G is a Sylow p-subgroup of X where p is the prime whose power is equal to the order of G. If $N_X(G) = G$, then since G is abelian it follows that $N_X(G) = C_X(G)$. So, by the *Burnside transfer theorem*, G has a normal p-complement in X, contradicting Theorem 2. Hence we have that $G < N_X(G)$. □

Example. Let $G := \langle a \rangle X \langle b \rangle$ where $a^2 = e = b^3$. Embed G into S_6 by the Cayley mapping. Let $\bar{G}$ be the isomorphic copy of G under this mapping. Then we have that

$S_6 = \mathrm{Sym}(\Omega)$ where $\Omega = \{e, a, b, b^2, ab, ab^2\}$. Denote by $1, 2, \ldots, 6$ the elements e, a, b, b^2, ab, ab^2 respectively. It is easy to check that a corresponds to the permutation (12)(35)(46) and that b corresponds to (134)(256). Let $c = (56)(34)$. We check that c coes not lie in $\bar{G}$ but $c^{-1}\bar{G}c = \bar{G}$.

Proposition 4. *Let G be a group satisfying the hypothesis of Theorem 2. If G is a Hall subgroup of X, then G is not contained in the centre of $N_X(G)$. In particular, if G is a Hall subgroup of X, then G is not abelian.*

Proof. If G is any finite group and H is a Hall subgroup of G such that H is contained in the centre of $N_G(H)$, then it can be shown that H has a normal complement in G (see for example, Kurzweil [6, p. 145]). So the proposition now follows using Theorem 2. □

Using Theorem 2 and [9, Theorem 1], we get the following

Corollary 5. *Let G be a group satisfying the hypothesis of Theorem 2. Assume that X is a a solvable group whose system normalizer is self-normalizing and that G is a Hall subgroup of X. Then we have that $G < N_G(X)$.*

Using Theorem 2 and Carter [4], we get the following

Corollary 6. *Let G be a nilpotent group satisfying the hypothesis of Theorem 2. If G is self-normalizing and Hall in the group X, then the Sylow subgroups of G are not regular in X.*

Finally, we include in the following proposition some properties of a group embedded in a symmetric group by the Cayley mapping:

Proposition 7. *Let G be a Π-group satisfying the hypothesis of Theorem 2. Then we have:*

(i) *The group X cannot be a Frobenius group with kernel G.*

(ii) *If G is a nilpotent, Hall, subgroup of X, then there exists at least two elements of G which are conjugate in X but not in G.*

(iii) *The group G cannot be a hyper-focal, Hall subgroup of X.*

Proof. (i) If X is a Frobenius group with kernel as G, then by a theorem of Frobenius (see for example, Huppert [5, Hauptsatz 7.6, p. 495]), G has a normal Π-complement which contradicts Theorem 2.

(ii) This follows from Theorem 2 using the result that if G is any group with a nilpotent, Hall subgroup H such that any two elements of H which are conjugate in G, are conjugate in H, then G has a normal Π-complement. (see for example, Passman [7, Corollary 12.5, p. 102]).

(iii) We recall the definition of a hyper-focal subgroup. If G is any finite group and $H \leq G$, define $\mathrm{Foc}_G(G)$ to be the subgroup generated by all the commutators $[h, g]$ with $h \in H$, $g \in G$ and $[h, g] \in H$. Define recursively $H_i := \mathrm{Foc}_G(H_{i-1})$. We say that H is hyperfocal in G if for some $n, H_n = (e)$. Now, if G is any group with a hyper-focal, Hall subgroup then G has a normal Π-complement (see for example, Passman [7, Theorem 12.4, p. 101]). Hence in our case, the result follows now using Theorem 2. □

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References

[1] B. Bauman, Endliche nicht auflösbare Gruppen mit einer nilpotenten minimalen Untergruppe, J. Algebra 38 (1976) 119–135.

[2] P. Bhattacharya, On the normaliser of a group in the Cayley representation, Bull. Australian Math. Soc. 25 (1982) 81–84.

[3] R.W. Carter, Nilpotent self-normaliziang subgroups and system normalizers, Proc. Lond. Math. Soc. 12 (1962) 535–563.

[4] R.W. Carter, Normal complements of nilpotent self-normalizing subgroups, Math. Z. 78 (1962) 149–150.

[5] B. Huppert, Endliche Gruppen I (Springer, New York, 1967).

[6] H. Kurzweil, Endliche Gruppen, Eine Einführung in die Theorie der endlichen Gruppen (Springer, Berlin, 1977).

[7] D.S. Passman, Permutation Groups (Benjamin, New York, 1968).

[8] J. Rose, On finite insoluble groups with nilpotent maximal subgroups, J. Algebra 48 (1977) 182–196.

[9] S.K. Sehgal and W.A. McWorter, Normal complements of Carter subgroups, Illinois J. Math. 12 (1968) 510–512.

[10] J. Thompson, A special class of nonsolvable groups, Math. Z. 72 (1960) 453–462.